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Parametric quintic spline for time fractional Burger's and coupled Burgers' equations

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Abstract

In this paper, the numerical solutions of time fractional Burger's and coupled Burgers' equations are obtained using the parametric quintic spline method with a local truncation error of order eight in distance direction. Additionally, the von Neumann method was utilized for studying the stability analysis of the present method. Finally, to show the accuracy of this method, some examples with different cases for Burger's and coupled Burgers' equations are presented and their results are compared with the previous methods.

Keywords: Parametric quintic spline method; Fractional Burger's equation; Fractional coupled Burger's equations; Von Neumann stability analysis

1 Introduction

Consider the time fractional Burger's equation (TFBE) [1–17] and the time fractional coupled Burgers' equations (TFCBEs) [18–25] defined as follows, respectively:

$$D_t^\alpha u(x,t) + u(x,t)D_x u(x,t) - sD_{xx}u(x,t) = f(x,t) \quad \text{for } 0 < \alpha < 1, \tag{1}$$

$$D_t^\alpha\big(u(x,t)\big) = D_{xx}\big(u(x,t)\big) + 2u(x,t)D_x\big(u(x,t)\big) - D_x\big(u(x,t)v(x,t)\big), \tag{2}$$

$$D_t^\beta\big(v(x,t)\big) = D_{xx}\big(v(x,t)\big) + 2v(x,t)D_x\big(v(x,t)\big) - D_x\big(u(x,t)v(x,t)\big), \tag{3}$$

where s is a viscosity parameter, $D_t^\alpha u(x,t)$ and $D_t^\beta v(x,t)$ are the Caputo fractional derivatives of orders α and β [4, 7–11, 15, 26–30] defined as follows:

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi}(t-\xi)^{-\alpha}\, d\xi \quad \text{for } 0 < \alpha < 1, \tag{4}$$

by the same manner, we can define $D_t^\beta v(x,t)$.

Firstly, many numerical techniques are used to obtain the solutions of equations (1)–(3) such as Adomian decomposition method (ADM) [1], variational iteration method (VIM) [2], cubic parametric spline (CPS) method [3], quadratic B-spline Galerkin method (QBSGM) [4], cubic trigonometric B-splines method (CTBSM) [7], Legendre–Galerkin spectral method (LGSM) [9], Crank–Nicolson approach (CNA) [14], finite difference method (FDM) [16], Chebyshev collocation method (CCM) [20], spectral collocation method (SCM) [22], etc. For more details, Momani [1] investigated a nonperturbative analytical solution of TFBE using ADM, while Inc [2] solved it using VIM. El-Danaf and Hadhoud [3] used the cubic parametric spline method to obtain the numerical solution of equation (1). Also, Esen and Tasbozan [4] and Yokus and Kaya [5] applied the quadratic B-spline Galerkin method [4] and the expansion method with the Cole–Hopf transformation [5] to get its solution. Furthermore, Hassani and Naraghirad [6] and Yaseen and Abbas [7] gave its solution through an optimization method based on the generalized polynomials [6] and CTBSM [7]. Alsaedi et al. [8] gave a smooth solution of TFBE, while Li et al. [9, 10] discussed its solution using LGSM [9] and the local discontinuous Galerkin method (LDGM) [10]. A numerical technique based on an extended cubic B-spline function was used to solve TFBE by Akram et al. [12] and Majeed et al. [13]. Also, Onal and Esen [14], Chen et al. [15], Yadav and Pandey [16], and Wang [17] approximated its solution using CNA [14], Fourier spectral method [15], FDM [16], and the separation of variables method [17]. The backward substitution method (BSM) and FDM were used by Safari and Chen [18] to obtain the solutions of TFCBEs. Doha et al. [19] and Albuohimad and Adibi [20] established their solutions by Jacobi–Gauss–Lobatto collocation method (JGLCM) [19] and CCM [20]. Ahmed et al. [21, 22] solved them using Laplace–Adomian decomposition method (LADM) [21], Laplace variational iteration method (LVIM) [21], and SCM [22]. Also, the generalized differential transform method [23] and the meshfree spectral method [24] have been used to solve TFCBEs by Liu and Hou [23] and Hussain et al. [24]. Furthermore, Bekir and Guner [31] solved equations (1)–(3) using the (G'/G)-expansion method, while Abdel-Salam and Hassan [32] solved equation (1) using the generalized exp-function method. Also, there are some references [33–38] about the numerical solutions of fractional equations, the researcher should read them.

Secondly, the nonpolynomial splines [26–30, 39–44] are used to solve many fractional order partial differential equations such as fractional subdiffusion problems [26, 27, 39, 40], fractional diffusion–wave problems [41, 42], fractional Schrödinger equation [28, 29], and fractional differential equations [30, 43]. One of the advantages of these methods is not only investigating an approximation for the function $u(x,t)$ but also for its derivatives. In this study, the parametric quintic spline method (PQSM) is used to investigate the numerical solutions of TFBE and TFCBEs with a local truncation error $O(h^8)$ in distance direction. Also, we demonstrate that the present method is stable and compare its results with the existing methods such as CPS [3], QBSGM [4], and BSM [18]. These comparisons conclude that the present method is more reliable and accurate.

This paper is organized as follows: In Sects. 2 and 3, the procedure of PQSM is presented in detail, then we apply it to solve TFBE and TFCBEs. Section 4 contains the stability analysis of the proposed method. Finally, the numerical results, the conclusion, and all abbreviations are given in Sects. 5 and 6, respectively.

2 Parametric quintic spline method

Firstly, suppose that $x_i = x_0 + ih$ are the nodes of a uniform partition of the interval $[a,b]$ with n subintervals, where $h = \frac{x_n - x_0}{n}$, $x_0 = a$, $x_n = b$, and $i = 0,1,2,\dots,n$. Also, consider that $t_j = jk$, where $j = 0,1,2,\dots,$ $t \in [0,T]$ and $k = \Delta t = t_{j+1} - t_j$, is the time step. Secondly, let

$Q_i(x,t_j)$ be the parametric quintic spline function defined as follows:

$$\begin{aligned} Q_i(x,t_j) &= a_i(t_j) + b_i(t_j)(x-x_i) + c_i(t_j)(x-x_i)^2 + d_i(t_j)(x-x_i)^3 \\ &\quad + e_i(t_j)\sinh\bigl(\tau(x-x_i)\bigr) + f_i(t_j)\cosh\bigl(\tau(x-x_i)\bigr) \\ &\quad \text{for } i = 0,1,2,\dots,n-1 \text{ and } x \in [x_i, x_{i+1}], \end{aligned} \tag{5}$$

where $a_i(t_j)$, $b_i(t_j)$, $c_i(t_j)$, $d_i(t_j)$, $e_i(t_j)$, and $f_i(t_j)$ are the spline coefficients and τ is a constant.

Suppose that $u_i^j = Q_i(x_i,t_j)$, $u_{i+1}^j = Q_i(x_{i+1},t_j)$, $R_i^j = Q_i''(x_i,t_j)$, $R_{i+1}^j = Q_i''(x_{i+1},t_j)$, $L_i^j = Q_i^{(4)}(x_i,t_j)$, $L_{i+1}^j = Q_i^{(4)}(x_{i+1},t_j)$, and $\Phi = h\tau$. Hence the spline coefficients can be determined as follows:

$$\begin{aligned} a_i(t_j) &= -\frac{L_i^j}{\tau^4} + u_i^j, \\ b_i(t_j) &= \frac{1}{6\Phi\tau^3}\bigl[2\bigl(3+\Phi^2\bigr)L_i^j + \bigl(-6+\Phi^2\bigr)L_{i+1}^j - \Phi^2\tau^2\bigl(2R_i^j + R_{i+1}^j\bigr) + 6\tau^4\bigl(u_{i+1}^j - u_i^j\bigr)\bigr], \\ c_i(t_j) &= \frac{1}{2}\left(-\frac{L_i^j}{\tau^2} + R_i^j\right), \\ d_i(t_j) &= -\frac{1}{6\Phi\tau}\bigl(L_{i+1}^j - L_i^j + \tau^2\bigl(R_i^j - R_{i+1}^j\bigr)\bigr), \\ e_i(t_j) &= \frac{1}{\tau^4}\bigl(-\coth(\Phi)L_i^j + \operatorname{csch}(\Phi)L_{i+1}^j\bigr), \end{aligned} \tag{6}$$

and $f_i(t_j) = \frac{1}{\tau^4}(L_i^j)$.

Using the continuity of 1st and 3rd derivatives of the spline function at (x_i,t_j), we have $Q_{i-1}'(x_i,t_j) = Q_i'(x_i,t_j)$ and $Q_{i-1}^{(3)}(x_i,t_j) = Q_i^{(3)}(x_i,t_j)$, then for $i = 1,2,3,\dots,n-1$, we get

$$R_{i-1}^j + 4R_i^j + R_{i+1}^j = \frac{6}{h^2}\bigl(u_{i-1}^j - 2u_i^j + u_{i+1}^j\bigr) - 6h^2\bigl(\gamma_1 L_{i-1}^j + 2\delta_1 L_i^j + \gamma_1 L_{i+1}^j\bigr), \tag{7}$$

$$R_{i-1}^j - 2R_i^j + R_{i+1}^j = h^2\bigl(\gamma L_{i-1}^j + 2\delta L_i^j + \gamma L_{i+1}^j\bigr), \tag{8}$$

where

$$\begin{aligned} \gamma &= \frac{1}{\Phi^2}\bigl(\Phi\operatorname{csch}(\Phi) - 1\bigr), \qquad \delta = \frac{1}{\Phi^2}\bigl(1 - \Phi\coth(\Phi)\bigr), \\ \gamma_1 &= \frac{-1}{\Phi^2}\left(\frac{1}{6} + \gamma\right) \quad \text{and} \quad \delta_1 = \frac{-1}{\Phi^2}\left(\frac{1}{3} + \delta\right). \end{aligned}$$

By multiplying equation (7) by γ and equation (8) by $-6\gamma_1$, then subtracting them, we get

$$\begin{aligned} L_i^j = \frac{1}{12h^2(\gamma_1\delta - \gamma\delta_1)}\biggl[&(\gamma + 6\gamma_1)R_{i-1}^j + 4(\gamma - 3\gamma_1)R_i^j + (\gamma + 6\gamma_1)R_{i+1}^j \\ &- \frac{6\gamma}{h^2}\bigl(u_{i-1}^j - 2u_i^j + u_{i+1}^j\bigr)\biggr], \end{aligned} \tag{9}$$

similarly, we can get L_{i-1}^j and L_{i+1}^j.

By substituting L_i^j, L_{i-1}^j, and L_{i+1}^j into equation (8), we get

$$\begin{aligned}
&(\gamma + 6\gamma_1)\big(R_{i-2}^j + R_{i+2}^j\big) + 2(2\gamma + \delta - 6\gamma_1 + 6\delta_1)\big(R_{i-1}^j + R_{i+1}^j\big) \\
&\quad + 2(\gamma + 4\delta + 6\gamma_1 - 12\delta_1)R_i^j \\
&\quad = \frac{6}{h^2}\big[\gamma\big(u_{i-2}^j + u_{i+2}^j\big) + 2(\delta - \gamma)\big(u_{i-1}^j + u_{i+1}^j\big) + 2(\gamma - 2\delta)u_i^j\big] \\
&\quad \text{for } i = 2, 3, 4, \dots, n-2.
\end{aligned} \tag{10}$$

The system of equations (10) gives $(n-3)$ equations in $(n-3)$ unknowns. Furthermore, to have a unique solution, we need two more equations which can be defined as follows:

$$\sum_{l=0}^{3} a_l u_l^j + h^2 \sum_{l=0}^{4} b_l R_l^j + t_1 = 0 \quad \text{for } i = 1, \tag{11}$$

$$\sum_{l=0}^{3} a_l u_{n-l}^j + h^2 \sum_{i=0}^{4} b_l R_{n-l}^j + t_{n-1} = 0 \quad \text{for } i = n-1. \tag{12}$$

By expanding equations (11) and (12) by Taylor's approximation about x_1 and x_{n-1}, respectively, we get

$$\begin{aligned}
&(a_0, a_1, a_2, a_3) = (0, 1, -2, 1), \\
&(b_0, b_1, b_2, b_3, b_4) = \left(\frac{1}{240}, -\frac{1}{10}, -\frac{97}{120}, -\frac{1}{10}, \frac{1}{240}\right), \\
&t_1 = \frac{31}{60{,}480}\big(h^8\big)u^{(8)}(x_1, t_j),
\end{aligned}$$

and $t_{n-1} = \frac{31}{60{,}480}(h^8)u^{(8)}(x_{n-1}, t_j)$.

The local truncation error (*LTE*) of PQSM can be established by expanding equation (10) by Taylor's approximation about x_i, then we have

$$\begin{aligned}
LTE_i &= \frac{1}{6}h^2(\gamma + \delta + 12\gamma_1 + 12\delta_1)u^{(4)}(x_i, t_j) \\
&\quad + \frac{1}{180}h^4(19\gamma + 4\delta + 210\gamma_1 + 30\delta_1)u^{(6)}(x_i, t_j) \\
&\quad + \frac{1}{30{,}240}h^6(571\gamma + 25\delta + 5208\gamma_1 + 168\delta_1)u^{(8)}(x_i, t_j) \\
&\quad + \frac{1}{907{,}200}h^8(1439\gamma + 14\delta + 11{,}430\gamma_1 + 90\delta_1)u^{(10)}(x_i, t_j) + \cdots.
\end{aligned} \tag{13}$$

By assuming $L_1 = \gamma + \delta + 12\gamma_1 + 12\delta_1$, $L_2 = 19\gamma + 4\delta + 210\gamma_1 + 30\delta_1$, and $L_3 = 571\gamma + 25\delta + 5208\gamma_1 + 168\delta_1$,, for $L_1 = L_2 = L_3 = 0$, we get

$$\gamma = \frac{31}{95}\delta, \qquad \gamma_1 = -\frac{109}{2850}\delta, \qquad \delta_1 = -\frac{103}{1425}\delta, \tag{14}$$

and $LTE_i = \frac{79}{1{,}795{,}500}\delta h^8 u^{(10)}(x_i, t_j) + O(h^{10})$, hence $LTE_i = O(h^8)$.

3 Applying PQSM on fractional Burgers' equations

To investigate the solutions of TFBE and TFCBEs using the PQSM method, we consider that the discretization of the fractional derivative equation (4) is defined as in [3] as follows:

$$\begin{aligned} &D_t^\alpha u(x,t) \\ &\quad = \frac{k^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \sum_{m=0}^{j-1} \bigl[\bigl((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}\bigr)\bigl(u(x_i,t_{m+1}) - u(x_i,t_m)\bigr)\bigr]. \end{aligned} \tag{15}$$

3.1 Time fractional Burger's equation

From equation (1), we have

$$R_i^j = (u_{xx})_i^j = \frac{1}{s}\bigl(D_t^\alpha U_i^j + U_i^j (U_x)_i^j - f(x_i,t_j)\bigr), \tag{16}$$

by using CNA, we can rewrite equation (16) as follows:

$$R_i^j = \frac{1}{s}\left(D_t^\alpha U_i^j + \frac{1}{2} U_i^j \bigl[(U_x)_i^j + (U_x)_i^{j-1}\bigr] - f(x_i,t_j)\right), \tag{17}$$

by substituting ($D_t^\alpha u$) from equation (15) into equation (17), we get

$$\begin{aligned} R_i^j = {}& \frac{1}{s}\left[\left(\frac{k^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\right) \sum_{m=0}^{j-1} \bigl[\bigl((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}\bigr)\bigl(U_i^{m+1} - U_i^m\bigr)\bigr]\right] \\ &+ \left(\frac{1}{2s}\right) U_i^j \bigl[(U_x)_i^j + (U_x)_i^{j-1}\bigr] - \left(\frac{1}{s}\right) f(x_i,t_j), \end{aligned} \tag{18}$$

where $i = 0,1,2,\dots,n$ and $j = 1,2,3,\dots,$ and

$$(U_x)_i^j = \begin{cases} \frac{1}{h}(U_{i+1}^j - U_i^j), & i = 0, \\ \frac{1}{2h}(U_{i+1}^j - U_{i-1}^j), & 1 \le i \le n-1, \\ \frac{1}{h}(U_i^j - U_{i-1}^j), & i = n. \end{cases}$$

By substituting R_i^j from equation (18) into equations (10)–(12) and solving this nonlinear system with the initial and boundary conditions using any numerical method, such as Newton–Raphson method, we get the solution of time fractional Burger's equation (1).

3.2 Time fractional coupled Burgers' equations

By using CNA, we can rewrite equations (2) and (3) as follows:

$$\begin{aligned} R_{1i}^j = {}& D_t^\alpha U_i^j - \bigl((UU_x)_i^j + (UU_x)_i^{j-1}\bigr) \\ &+ \frac{1}{2}\bigl[\bigl((UV_x)_i^j + (UV_x)_i^{j-1}\bigr) + \bigl((VU_x)_i^j + (VU_x)_i^{j-1}\bigr)\bigr], \end{aligned} \tag{19}$$

$$\begin{aligned} R_{2i}^j = {}& D_t^\beta V_i^j - \bigl((VV_x)_i^j + (VV_x)_i^{j-1}\bigr) \\ &+ \frac{1}{2}\bigl[\bigl((UV_x)_i^j + (UV_x)_i^{j-1}\bigr) + \bigl((VU_x)_i^j + (VU_x)_i^{j-1}\bigr)\bigr], \end{aligned} \tag{20}$$

where $R_{1i}^j = (u_{xx})_i^j$ and $R_{2i}^j = (v_{xx})_i^j$.

The nonlinear terms in equations (19) and (20) can be defined as in [45] as follows:

$$(AB_x)_i^j = A_i^j B_{xi}^j,$$
$$(AB_x)_i^{j-1} = A_i^j B_{xi}^{j-1} + A_i^{j-1} B_{xi}^j - A_i^j B_{xi}^j, \tag{21}$$

where the symbols A and B in equation (21) can be used for U or V.

From equation (21) into equations (19) and (20) and by substituting $D_t^\alpha U_i^j$ and $D_t^\beta V_i^j$ from equation (15), we get

$$R_{1i}^j = \left(\frac{k^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\right)\sum_{m=0}^{j-1}\left[\left((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}\right)\left(U_i^{m+1} - U_i^m\right)\right]$$
$$- \left[U_i^{j-1}U_{xi}^j + U_i^j U_{xi}^{j-1}\right] + \frac{1}{2}\left[U_i^{j-1}V_{xi}^j + U_i^j V_{xi}^{j-1} + V_i^{j-1}U_{xi}^j + V_i^j U_{xi}^{j-1}\right], \tag{22}$$

$$R_{2i}^j = \left(\frac{k^{-\beta}}{(1-\beta)\Gamma(1-\beta)}\right)\sum_{m=0}^{j-1}\left[\left((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}\right)\left(V_i^{m+1} - V_i^m\right)\right]$$
$$- \left[V_i^{j-1}V_{xi}^j + V_i^j V_{xi}^{j-1}\right] + \frac{1}{2}\left[U_i^{j-1}V_{xi}^j + U_i^j V_{xi}^{j-1} + V_i^{j-1}U_{xi}^j + V_i^j U_{xi}^{j-1}\right], \tag{23}$$

where $i = 0, 1, 2, \dots, n$ and $j = 1, 2, 3, \dots,$ and

$$C_{xi}^j = \begin{cases} \frac{1}{h}(C_{i+1}^j - C_i^j), & i = 0, \\ \frac{1}{2h}(C_{i+1}^j - C_{i-1}^j), & 1 \le i \le n-1, \\ \frac{1}{h}(C_i^j - C_{i-1}^j), & i = n, \end{cases} \tag{24}$$

where the symbol C in equation (24) can be used for U or V.

Replacing R_i^j in equations (10)–(12) by R_{1i}^j once and R_{2i}^j again, then substituting R_{1i}^j and R_{2i}^j from equations (22) and (23),, we can solve this nonlinear system with the initial and boundary conditions using the Newton–Raphson method to obtain the solutions of time fractional coupled Burgers' equations (2) and (3).

4 Von Neumann stability analysis

In this section, we analyze the stability of PQSM using von Neumann method. For this purpose, firstly, we need to linearize the nonlinear terms in equation (18) as in [3] as follows:

$$R_i^j = \sum_{m=0}^{j-1}\left[\frac{k^{-\alpha}}{s(1-\alpha)\Gamma(1-\alpha)}\left((j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}\right)\left(U_i^{m+1} - U_i^m\right)\right]$$
$$+ \left(\frac{\varphi_1}{2sh}\right)\left[U_{i+1}^j - U_{i-1}^j + U_{i+1}^{j-1} - U_{i-1}^{j-1}\right] \quad \text{for } 1 \le i \le n-1 \text{ and } j \ge 1, \tag{25}$$

where $\varphi_1 = U_i^j$ is a local constant, and for simplicity we assume $f(x_i, t_j) = 0$.

Secondly, we consider that U_i^j is defined as follows:

$$U_i^j = \xi^j e^{l(kih)}, \tag{26}$$

where $l = \sqrt{-1}$ and ξ is the growth factor.

Substituting U_i^j from equation (26) into (25), we get

$$R_i^j = e^{l(kih)}\left[\sum_{m=0}^{j-1}\varphi_2\left[\Psi_{j,m}^{\alpha}\left(\xi^{m+1}-\xi^m\right)\right]+\left(\frac{\varphi_1}{2sh}\right)\left(\xi^j+\xi^{j-1}\right)\left(e^{l(kh)}-e^{-l(kh)}\right)\right]$$
$$\text{for } 0 \le i \le n \text{ and } j \ge 1, \tag{27}$$

where $\Psi_{j,m}^{\alpha} = (j-m)^{1-\alpha} - (j-m-1)^{1-\alpha}$ and $\varphi_2 = \frac{k^{-\alpha}}{s(1-\alpha)\Gamma(1-\alpha)}$.

Substituting U_i^j and R_i^j from equations (26) and (27) into equation (10), we get

$$\begin{aligned}&\left[\sum_{m=0}^{j-1}\varphi_2\left[\Psi_{j,m}^{\alpha}\left(\xi^{m+1}-\xi^m\right)\right]+\left(\frac{\varphi_1}{2sh}\right)\left(\xi^j+\xi^{j-1}\right)\left(e^{l(kh)}-e^{-l(kh)}\right)\right]\\&\quad\times\left[\rho_1\left(e^{l(-2kh)}+e^{l(2kh)}\right)+\rho_2\left(e^{l(-kh)}+e^{l(kh)}\right)+\rho_3\right]\\&\quad=\frac{6}{h^2}\xi^j\left[\gamma\left(e^{l(-2kh)}+e^{l(2kh)}\right)+2(\delta-\gamma)\left(e^{l(-kh)}+e^{l(kh)}\right)+2(\gamma-2\delta)\right]\\&\quad\text{for } j=1,2,3,\ldots,\end{aligned}\tag{28}$$

where $\rho_1 = \gamma + 6\gamma_1$, $\rho_2 = 2(2\gamma + \delta - 6\gamma_1 + 6\delta_1)$, and $\rho_3 = 2(\gamma + 4\delta + 6\gamma_1 - 12\delta_1)$.

Since $e^{l(-kh)} + e^{l(kh)} = 2\cos(kh)$ and $e^{l(kh)} - e^{-l(kh)} = 2l\sin(kh)$, then we have

$$\begin{aligned}&\left[\sum_{m=0}^{j-1}\varphi_2\left[\Psi_{j,m}^{\alpha}\left(\xi^{m+1}-\xi^m\right)\right]+l\left(\frac{\varphi_1}{sh}\right)\left(\xi^j+\xi^{j-1}\right)\sin(kh)\right]\\&\quad\times\left[2\rho_1\cos(2kh)+2\rho_2\cos(kh)+\rho_3\right]\\&\quad=\frac{6}{h^2}\xi^j\left[2\gamma\cos(2kh)+4(\delta-\gamma)\cos(kh)+2(\gamma-2\delta)\right]\\&\quad\text{for } j=1,2,3,\ldots.\end{aligned}\tag{29}$$

Putting $j = 1$ in equation (29), we have

$$\begin{aligned}&\left[\left(\varphi_2\Psi_{1,0}^{\alpha}+l\left(\frac{\varphi_1}{sh}\right)\sin(kh)\right)\left(2\rho_1\cos(2kh)+2\rho_2\cos(kh)+\rho_3\right)\right.\\&\quad\left.-\frac{6}{h^2}\left(2\gamma\cos(2kh)+4(\delta-\gamma)\cos(kh)+2(\gamma-2\delta)\right)\right]\xi^1\\&\quad=\left[\left(\varphi_2\Psi_{1,0}^{\alpha}-l\left(\frac{\varphi_1}{\nu h}\right)\sin(kh)\right)\left(2\rho_1\cos(2kh)+2\rho_2\cos(kh)+\rho_3\right)\right]\xi^0,\end{aligned}\tag{30}$$

hence

$$\left|\frac{\xi^1}{\xi^0}\right| = \left|\frac{(\varphi_2 - l\omega_1)\varphi_3}{(\varphi_2 + l\omega_1)\varphi_3 - \varphi_4}\right|,\tag{31}$$

where $\Psi_{1,0}^{\alpha} = (1)^{1-\alpha} = 1$, $\omega_1 = (\frac{\varphi_1}{sh})\sin(\theta)$, $\varphi_3 = (2\rho_1\cos(2\theta) + 2\rho_2\cos(\theta) + \rho_3)$, $\varphi_4 = \frac{6}{h^2}(2\gamma \times \cos(2\theta) + 4(\delta-\gamma)\cos(\theta) + 2(\gamma - 2\delta))$, and $\theta = kh$.

Now, we can study the worth cases for equation (31), when $\cos(\theta) = \pm 1$, as follows:

(i) For $\theta = 2\pi n$ and $n = 0, \pm 1, \pm 2, \dots$, we get $\omega_1 = 0$, $\varphi_3 = 12(\gamma + \delta)$, and $\varphi_4 = 0$, hence $|\frac{\xi^1}{\xi^0}| = 1$.

(ii) For $\theta = \pi n$ and $n = \pm 1, \pm 3, \pm 5, \dots$, we get $\omega_1 = 0$, $\varphi_3 = 4(-\gamma + \delta + 12\gamma_1 - 12\delta_1)$, and $\varphi_4 = -\frac{48}{h^2}(\delta - \gamma)$, hence $|\frac{\xi^1}{\xi^0}| = \frac{\varphi_2\varphi_3}{\varphi_2\varphi_3 + \frac{48}{h^2}(\delta-\gamma)} < 1$ for $\delta - \gamma > 0$,

therefore,

$$\left|\frac{\xi^1}{\xi^0}\right| \le 1 \quad \text{and} \quad \delta > \gamma. \tag{32}$$

Similarly, putting $j = 2$ in equation (29), we get

$$\bigl[\varphi_2\bigl(\Psi^{\alpha}_{2,0}(\xi^1 - \xi^0)\bigr) + \varphi_2\bigl(\Psi^{\alpha}_{2,1}(\xi^2 - \xi^1)\bigr) + l\omega_1(\xi^2 + \xi^1)\bigr]\varphi_3 = (\varphi_4)\xi^2. \tag{33}$$

Since $\Psi^{\alpha}_{2,0} = ((2)^{1-\alpha} - 1)$, $\Psi^{\alpha}_{2,1} = 1$ and from equation (32) we can take $\xi^1 = \xi^0$ and $\delta > \gamma$, then we have

$$\bigl[\varphi_2(\xi^2 - \xi^0) + l\omega_1(\xi^2 + \xi^0)\bigr]\varphi_3 = (\varphi_4)\xi^2, \tag{34}$$

hence

$$\left|\frac{\xi^2}{\xi^0}\right| = \left|\frac{(\varphi_2 - l\omega_1)\varphi_3}{(\varphi_2 + l\omega_1)\varphi_3 - \varphi_4}\right| \le 1. \tag{35}$$

In general, for $j = 1, 2, 3, \dots$, we have $|\frac{\xi^j}{\xi^0}| \le 1$ and $\delta > \gamma$. Thus, the proposed method is stable.

5 Numerical results

In this section, the numerical solutions of TFBE and TFCBEs are obtained by using the PQSM method as introduced in Sects. 2 and 3. In addition, we discuss their solutions in two cases according to the spline parameters given by equation (14) as follows:

Case 1: As in [26, 29, 42], if we take $\gamma + \delta = \frac{1}{2}$, then we obtain

$$\begin{aligned} &\gamma = \frac{31}{252}, \qquad \delta = \frac{95}{252}, \qquad \gamma_1 = -\frac{109}{7560}, \\ &\delta_1 = -\frac{103}{3780} \quad \text{and} \quad LTE = \frac{79}{4{,}762{,}800}h^8. \end{aligned}$$

Case 2: Let $\delta = 0.001$, hence we have

$$\begin{aligned} &\gamma = \frac{31}{95{,}000}, \qquad \gamma_1 = -\frac{109}{2{,}850{,}000}, \\ &\delta_1 = -\frac{103}{1{,}425{,}000} \quad \text{and} \quad LTE = \frac{79}{1{,}795{,}500{,}000}h^8. \end{aligned}$$

To show the accuracy of the proposed method, we obtain the absolute errors or the error norms L_2 and L_∞ in each example. The error norms are demonstrated as follows:

$$L_\infty = \operatorname*{Max}_{1 \le i \le n-1} \bigl|(u_{\text{Exact}})^j_i - U^j_i\bigr|, \tag{36}$$

$$L_2 = \sqrt{h\sum_{i=1}^{n-1}\left((u_{\text{Exact}})_i^j - U_i^j\right)^2}. \tag{37}$$

Problem 5.1 Consider the time fractional Burger's equation (1) [4] with

$$f(x,t) = \frac{2}{\Gamma(3-\alpha)}\left(t^{2-\alpha}e^x\right) + t^4e^{2x} - s\left(t^2e^x\right), \tag{38}$$

the initial condition is

$$u(x,0) = 0 \quad \text{for } 0 \le x \le 1, \tag{39}$$

the boundary conditions are

$$u(0,t) = t^2 \quad \text{and} \quad u(1,t) = et^2, \tag{40}$$

and its exact solution is

$$u(x,t) = t^2e^x. \tag{41}$$

Table 1 contains the exact and approximated solutions of Problem 5.1 by using PQSM with $k = 0.002$ and QBSGM [4] with $k = 0.00025$ at $t = 1, s = 1, \alpha = 0.5$, and $h = 0.05, 0.025,$ and 0.0125. Also, Table 2 illustrates the numerical results for $t = 1, s = 1, \alpha = 0.5, n = 80$ and

Table 1 The numerical solutions and error norms of Problem 5.1 for $s = 1$, $\alpha = 0.5$, and $t = 1$

x	PQSM ($k = 0.002$)			QBSGM [4] ($k = 0.00025$)			Exact
	$n = 20$	$n = 40$	$n = 80$	$n = 20$	$n = 40$	$n = 80$	
0.1	1.105145	1.105166	1.105171	1.105287	1.105216	1.105197	1.105170918
0.2	1.221357	1.221394	1.221403	1.221644	1.221493	1.221455	1.221402758
0.3	1.349794	1.349846	1.349860	1.350217	1.349992	1.349935	1.349858808
0.4	1.491745	1.491809	1.491825	1.492287	1.491996	1.491922	1.491824698
0.5	1.648629	1.648703	1.648722	1.649270	1.648922	1.648838	1.648721271
0.6	1.822021	1.822099	1.822119	1.822727	1.822342	1.822247	1.822118800
0.7	2.013657	2.013733	2.013753	2.014378	2.013979	2.013882	2.013752707
0.8	2.225462	2.225523	2.225541	2.226118	2.225747	2.225661	2.225540928
0.9	2.459564	2.459593	2.459603	2.460020	2.459745	2.459680	2.459603111
L_2	3.11×10^{-4}	9.06×10^{-5}	3.90×10^{-6}	8.48×10^{-4}	1.62×10^{-4}	9.26×10^{-5}	
L_∞	9.83×10^{-5}	2.04×10^{-5}	7.16×10^{-7}	6.25×10^{-4}	2.27×10^{-4}	1.33×10^{-4}	

Table 2 The numerical solutions and error norms of Problem 5.1 for $s = 1$, $\alpha = 0.5$, $n = 80$, and $t = 1$

x	PQSM	QBSGM [4]		
	$k = 0.002$	$k = 0.002$	$k = 0.001$	$k = 0.00025$
0.1	1.105171	1.105356	1.105276	1.105216
0.2	1.221403	1.221768	1.221611	1.221493
0.3	1.349860	1.350395	1.350164	1.349992
0.4	1.491825	1.492516	1.492218	1.491996
0.5	1.648722	1.649543	1.649188	1.648922
0.6	1.822119	1.823031	1.822636	1.822342
0.7	2.013753	2.014687	2.014282	2.013979
0.8	2.225541	2.226387	2.226020	2.225747
0.9	2.459603	2.460180	2.459931	2.459745
L_2	3.90×10^{-6}	6.61×10^{-4}	3.75×10^{-4}	9.26×10^{-5}
L_∞	7.16×10^{-7}	9.37×10^{-4}	5.30×10^{-4}	1.33×10^{-4}

Table 3 The numerical solutions and error norms of Problem 5.1 for $s = 1$, $n = 40$, and $t = 1$

x	PQSM ($k = 0.002$)		QBSGM [4] ($k = 0.00025$)	
	$\alpha = 0.25$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.75$
0.1	1.105164	1.105179	1.105217	1.105216
0.2	1.221390	1.221418	1.221495	1.221493
0.3	1.349841	1.349880	1.349995	1.349990
0.4	1.491802	1.491850	1.492000	1.491993
0.5	1.648696	1.648749	1.648928	1.648920
0.6	1.822091	1.822147	1.822348	1.822339
0.7	2.013725	2.013779	2.013984	2.013977
0.8	2.225517	2.225563	2.225750	2.225744
0.9	2.459589	2.459618	2.459747	2.459744
L_2	1.26×10^{-4}	1.33×10^{-4}	1.65×10^{-4}	1.60×10^{-4}
L_∞	2.80×10^{-5}	2.85×10^{-5}	2.33×10^{-4}	2.25×10^{-4}

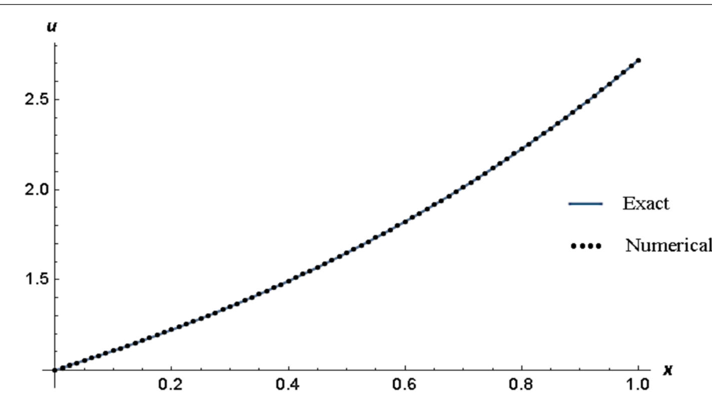

Figure 1 The exact solution and the numerical solution using PQSM of Problem 5.1 for $h = 0.0125$, $k = 0.002$, $\alpha = 0.5$, and $t = 1$

different values of k. These results show that the L_∞ error norm at $n = 80$ for PQSM with $k = 0.002$ is $O(10^{-7})$, while it is $O(10^{-4})$ for QBSGM [4] with $k = 0.00025$, hence PQSM is better than QBSGM [4]. Moreover, the numerical results for $\alpha = 0.25$, $\alpha = 0.75$, $s = 1$, $h = 0.025$, and $t = 1$ are shown in Table 3. Figure 1 shows the relation between the exact solution and the PQSM solution. Also, the approximated solutions and the absolute errors behavior for $\alpha = 0.5$ and 0.75 are given in Figs. 2 and 3. These computations are obtained according to *Case* 1.

Problem 5.2 Consider the time fractional Burger's equation (1) [3] with $f(x,t) = 0$, the initial condition is

$$u(x,0) = \frac{\mu + \gamma + (\gamma - \mu)\operatorname{Exp}[\frac{\mu}{s}(x-\xi)]}{1 + \operatorname{Exp}[\frac{\mu}{s}(x-\xi)]} \quad \text{for } -3 \le x \le 3, \tag{42}$$

and the boundary conditions are given as follows:

(i) For $\alpha = 1$,

$$u(-3,t) = \frac{\mu + \gamma + (\gamma - \mu)\operatorname{Exp}[\frac{\mu}{s}(-3 - \gamma t - \xi)]}{1 + \operatorname{Exp}[\frac{\mu}{s}(-3 - \gamma t - \xi)]} \tag{43}$$

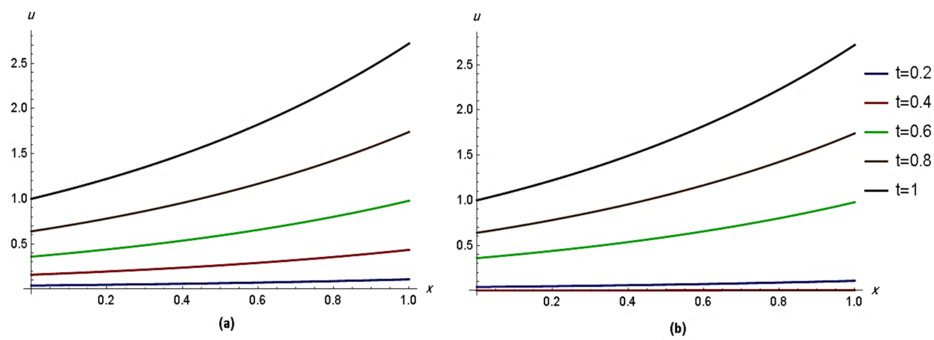

Figure 2 The approximated solutions of Problem 5.1 using PQSM for $h = 0.025$, $k = 0.002$, (**a**) $\alpha = 0.5$, and (**b**) $\alpha = 0.75$

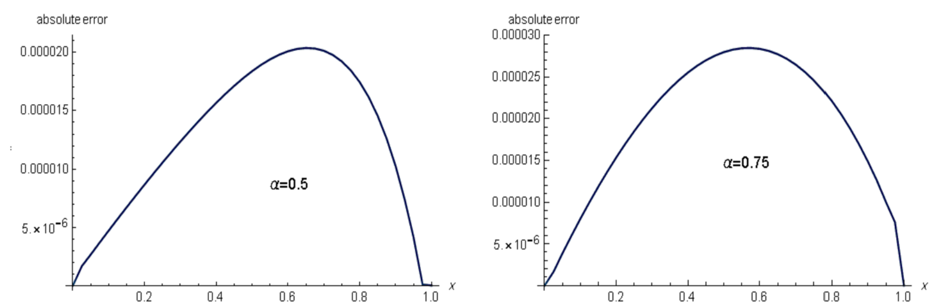

Figure 3 The absolute errors distribution of Problem 5.1 for $h = 0.025$, $k = 0.002$, and $t = 1$

and

$$u(3,t) = \frac{\mu + \gamma + (\gamma - \mu)\,\mathrm{Exp}[\frac{\mu}{s}(3 - \gamma t - \xi)]}{1 + \mathrm{Exp}[\frac{\mu}{s}(3 - \gamma t - \xi)]}; \tag{44}$$

(ii) For $0 < \alpha < 1$,

$$u(-3,t) \cong 0.699993 + \left(1.07 \times 10^{-5}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \left(9.67 \times 10^{-6}\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left(1.16 \times 10^{-5}\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \tag{45}$$

and

$$u(3,t) \cong 0.100815 + \left(1.3 \times 10^{-3}\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \left(1.17 \times 10^{-3}\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \left(5.72 \times 10^{-6}\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}. \tag{46}$$

The exact solution is

$$u(x,t) = \frac{\mu + \gamma + (\gamma - \mu)\,\mathrm{Exp}[\frac{\mu}{s}(x - \gamma t - \xi)]}{1 + \mathrm{Exp}[\frac{\mu}{s}(x - \gamma t - \xi)]} \quad \text{for } \alpha = 1. \tag{47}$$

Table 4 The absolute errors of Problem 5.2 for $\alpha = 1$, $s = 0.1$, $\mu = 0.3$, $\gamma = 0.4$, and $\xi = 0.8$

t	PQSM	CPS [3]
1.0	4.346×10^{-4}	4.632×10^{-3}
2.0	6.528×10^{-4}	5.267×10^{-3}
2.5	7.264×10^{-4}	5.569×10^{-3}
3.0	7.840×10^{-4}	5.857×10^{-3}

Table 5 The numerical solutions and the absolute errors of Problem 5.2 for $t = 3$, $\alpha = 1$, $s = 0.1$, $\mu = 0.3$, $\gamma = 0.4$, and $\xi = 0.8$

x	Exact solution	PQSM	CPS [3]	Absolute errors of	
				PQSM	CPS [3]
−1.80	0.6999932828	0.69999336	0.69999302	7.614×10^{-8}	2.608×10^{-7}
−1.50	0.6999834786	0.69998367	0.69998276	1.938×10^{-7}	7.193×10^{-7}
−0.96	0.6999165251	0.69991749	0.69991259	9.657×10^{-7}	3.931×10^{-6}
−0.48	0.6989641713	0.69897337	0.69963109	9.198×10^{-6}	6.669×10^{-4}
0.00	0.6985164261	0.69852846	0.69844644	1.203×10^{-5}	6.998×10^{-5}
0.48	0.6937877575	0.69381151	0.69349918	2.375×10^{-5}	2.886×10^{-4}
0.96	0.6746261369	0.67458239	0.67349066	4.375×10^{-5}	1.135×10^{-3}
1.50	0.5905446857	0.58997244	0.58651442	5.722×10^{-4}	4.030×10^{-3}
1.80	0.4873937837	0.48661686	0.48167270	7.769×10^{-4}	5.721×10^{-3}

Table 6 The numerical solutions of Problem 5.2 for $t = 2$, $s = 0.1$, $\mu = 0.3$, $\gamma = 0.4$, and $\xi = 0.8$

x	$\alpha = 0.2$		$\alpha = 0.8$	
	PQSM	CPS [3]	PQSM	CPS [3]
−1.80	0.69990818	0.69990801	0.69995843	0.69995837
−1.50	0.69976764	0.69976704	0.69989353	0.69989305
−0.96	0.69881774	0.69881444	0.69945724	0.69945423
−0.48	0.68593695	0.69505832	0.69348579	0.69771999
0.00	0.68023291	0.68019612	0.69080108	0.69075808
0.48	0.62975137	0.62969696	0.66570607	0.66557967
0.96	0.50954256	0.50952078	0.59278878	0.59246071
1.50	0.32247745	0.32242313	0.42059669	0.41986080
1.80	0.23615545	0.23607121	0.30646604	0.30566973

The approximated solutions of Problem 5.2 using PQSM and CPS [3] are given in Tables 4–6 with the following conditions: $h = k = 0.01$, $s = 0.1$, $\mu = 0.3$, $\gamma = 0.4$, $\xi = 0.8$, and $\alpha = 0.2, 0.8$, and 1. These results indicate that PQSM is the most accurate. Table 4 shows that the maximum absolute error of PQSM is $O(h^{-4})$, while it is $O(h^{-3})$ for CPS [3]. The exact solution and the numerical solution using PQSM for different values of α are given in Figs. 4, 5 and 6. These results are determined according to *Case* 2.

Problem 5.3 Consider the time fractional coupled Burgers' equations (2) and (3) [18] with the initial conditions:

$$u(x,0) = v(x,0) = \sin(x) \quad \text{for } 0 \le x \le 1, \tag{48}$$

the boundary conditions are

$$u(0,t) = v(0,t) = 0 \quad \text{and} \quad u(1,t) = v(1,t) = \sin(1)e^{-t}, \tag{49}$$

and their exact solutions are

$$u(x,t) = v(x,t) = \sin(x)e^{-t}. \tag{50}$$

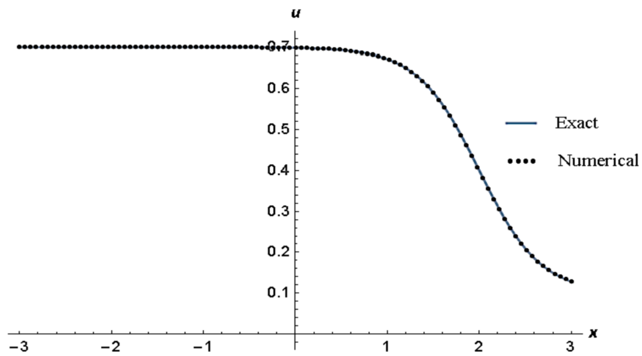

Figure 4 The exact solution and the numerical solution using PQSM of Problem 5.2 for $h = k = 0.01$, $\alpha = 1$, and $t = 3$

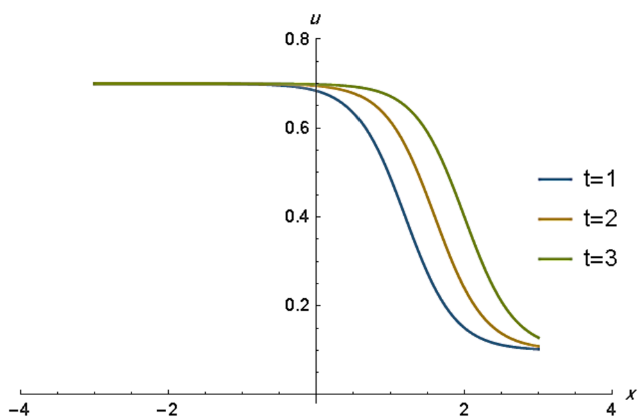

Figure 5 The approximated solutions of Problem 5.2 using PQSM for $\alpha = 1$, $h = k = 0.01$, and $t = 1, 2$, and 3

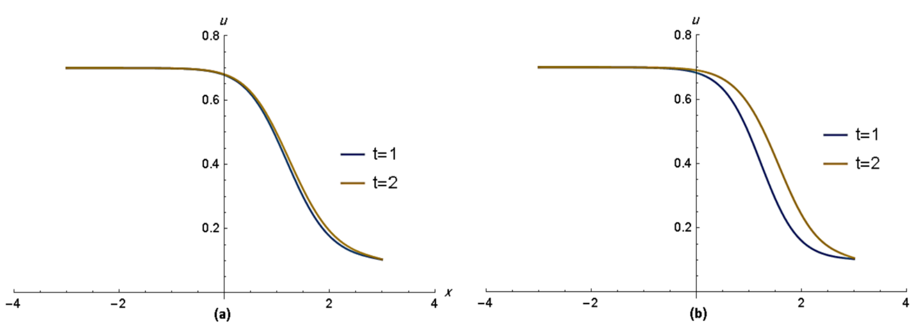

Figure 6 The approximated solutions of Problem 5.2 using PQSM for $h = k = 0.01$, (**a**) $\alpha = 0.2$ and (**b**) $\alpha = 0.8$

Table 7 illustrates the exact solution and the numerical solution of Problem 5.3 given by PQSM for $\alpha = \beta = 1$ and at time $t = 1$. Comparisons between PQSM and BSM [18] are given in Table 8 for $\alpha = \beta = 0.6, 0.9$, and 1, and they conclude that PQSM gave better results than BSM [18]. Also, the approximated solutions for $\alpha = \beta = 0.2, 0.6$, and 0.9 are shown in Table 9. The solution of Problem 5.3 and the absolute errors are shown in Figs. 7 and 8. These computations are investigated according to *Cases* 1 and 2.

Table 7 The numerical solution and the absolute errors of Problem 5.3 for $t = 1$, $h = 0.1$, $k = 0.01$, and $\alpha = \beta = 1$

x	Exact	PQSM	Absolute errors
0.1	0.036726662	0.0367541	2.749×10^{-5}
0.2	0.073086362	0.0731449	5.859×10^{-5}
0.3	0.108715808	0.1088009	8.510×10^{-5}
0.4	0.143259002	0.1433644	1.054×10^{-4}
0.5	0.176370799	0.1764883	1.175×10^{-4}
0.6	0.207720358	0.2078400	1.197×10^{-4}
0.7	0.236994443	0.2371047	1.103×10^{-4}
0.8	0.263900558	0.2639884	8.788×10^{-5}
0.9	0.288169866	0.2882213	5.141×10^{-5}

Table 8 The L_2 error norm of Problem 5.3 for $k = 0.01$

t	$\alpha = \beta = 1$		$\alpha = \beta = 0.9$		$\alpha = \beta = 0.6$	
	PQSM	BSM [18]	PQSM	BSM [18]	PQSM	BSM [18]
0.01	6.48×10^{-5}	4.16×10^{-2}	6.17×10^{-3}	2.68×10^{-2}	4.02×10^{-2}	2.49×10^{-2}
0.05	2.43×10^{-4}	3.99×10^{-2}	1.57×10^{-2}	3.01×10^{-2}	6.31×10^{-2}	2.58×10^{-2}
0.10	3.72×10^{-4}	3.79×10^{-2}	1.74×10^{-2}	2.88×10^{-2}	5.82×10^{-2}	2.31×10^{-2}
0.50	4.31×10^{-4}	2.55×10^{-2}	2.80×10^{-4}	1.64×10^{-2}	5.23×10^{-3}	5.46×10^{-3}
1.00	2.71×10^{-4}	1.54×10^{-2}	8.13×10^{-3}	7.87×10^{-3}	2.41×10^{-2}	4.84×10^{-3}

Table 9 The numerical solution of Problem 5.3 for $t = 1$, $h = 0.1$, and $k = 0.01$

x	$\alpha = \beta = 0.2$	$\alpha = \beta = 0.6$	$\alpha = \beta = 0.9$
0.1	0.0421081	0.0394294	0.0376431
0.2	0.0833382	0.0783387	0.0748691
0.3	0.1230931	0.1161984	0.1112536
0.4	0.1607006	0.1524908	0.1463852
0.5	0.1955216	0.1867052	0.1798625
0.6	0.2269384	0.2183411	0.2112986
0.7	0.2543721	0.2469104	0.2403234
0.8	0.2772507	0.2719389	0.2665878
0.9	0.2949838	0.2929689	0.2897663

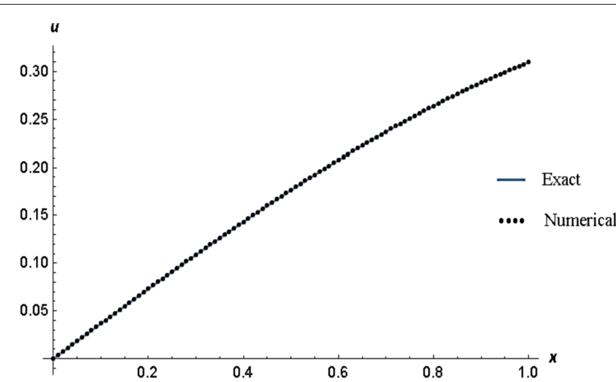

Figure 7 The exact solution and the numerical solution using PQSM of Problem 5.3 for $h = k = 0.01$, $\alpha = \beta = 1$, and $t = 1$

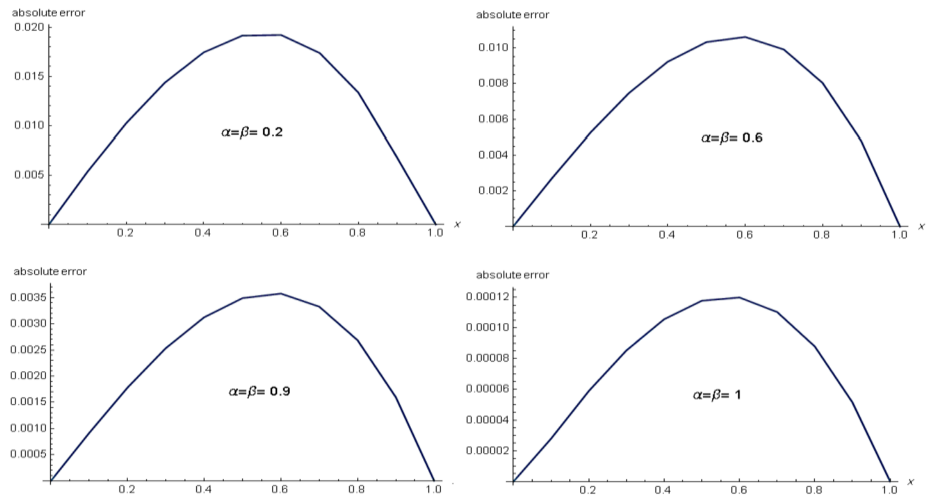

Figure 8 The absolute errors distribution of Problem 5.3 using PQSM for $h = 0.1$, $k = 0.01$, $t = 1$, and $\alpha = 0.2, 0.6, 0.9$, and 1

6 Conclusion

In this work, the solutions of TFBE and TFCBEs have been investigated using PQSM whose local truncation error is $O(h^8)$. We showed that the proposed method is stable. In addition, the given results are obtained for different values of the fractional order α and compared with the previous methods, which verified that the present method has good accuracy and efficiency.

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Abbreviations
TFBE, Time fractional Burger's equation; TFCBEs, Time fractional coupled Burgers' equations; PQSM, Parametric quintic spline method; CPS, Cubic parametric spline; QBSGM, Quadratic B-spline Galerkin method; BSM, Backward substitution method; ADM, Adomian decomposition method; VIM, Variational iteration method; CTBSM, Cubic trigonometric B-spline method; LGSM, Legendre–Galerkin spectral method; CNA, Crank–Nicolson approach; FDM, Finite difference method; CCM, Chebyshev collocation method; SCM, Spectral collocation method; LDGM, Local discontinuous Galerkin method; JGLCM, Jacobi–Gauss–Lobatto collocation method; LADM, Laplace–Adomian decomposition method; LVIM, Laplace variational iteration method; LTE, Local truncation error.

Availability of data and materials
Not applicable.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Applicable.

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Author contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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